Hyers-Ulam Stability for Linear Operators on Banach Spaces

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Outline of the talk

In 1940, in a talk before the Mathematics club of University of Winsconsin (US), S.M. Ulam proposed the first stability problem concerning group homomorphisms. In 1941, Hyers solved the problem in the framework of Banach spaces. In 1978, Th.M. Rassias investigated the stability of the linear mapping when the norm of the Cauchy difference is bounded by an expression. Since then several mathematicians were attracted to this result of Rassias and investigated a number of stability problems of functional equations. This stability phenomenon that was introduced and proved by Th.M. Rassias in his 1978 paper is called the Hyers-Ulam-Rassias stability. T. Miura at el. in 2003 introduced the notion of the Hyers-Ulam stability of a mapping (not necessarily linear) between two normed linear spaces. In this lecture, we discuss Hyers-Ulam stability for bounded / closed linear operators on Banach / Hilbert spaces.

Introduction

The function $f : \mathbb{R} \to \mathbb{R}$ is said to be **additive** if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

The above equation is called **Cauchy functional equation.**

The solutions of the linear functional equation $f(x+y) = f(x) + f(y)$ have been investigated for many spaces.

Additive Function

Exercise 1.

Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function. Prove the folllowing :

1. $f(0) = 0$.
2. $f(-x) = -f(x)$.
3. $f(nx) = nf(x),\ n \in \mathbb{Z}$.
4. $f(\frac{m}{n}x) = \frac{m}{n}f(x),\ m, n \in \mathbb{Z}$.
5. $f(r) = cr$, *for all* $r \in \mathbb{Q}$, *for some* $c \in \mathbb{R}$.

Theorem 2.

If the additive function $f : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$f(x) = c\ x \quad \text{for all } x \in \mathbb{R}$$

for some $c \in \mathbb{R}$.

Additive Function

Theorem 3.

Let f be an additive function satisfying $f(x) = cx$ with $c = f(1) > 0$. Then the following conditions are equivalent:

1. *f is continuous at a point x_0.*
2. *f is monotonically increasing.*
3. *f is non-negative for non-negative x.*
4. *f is bounded above on a finite interval.*
5. *f is bounded below on a finite interval.*
6. *f is bounded above (below) on a bounded set of positive Lebesgue measure.*
7. *f is bounded on a finite interval.*
8. *$f(x) = cx$, for all $x \in \mathbb{R}$, for some $c \in \mathbb{R}$.*
9. *f is locally Lebesgue integrable.*
10. *f is differentiable.*
11. *f is Lebesgue measurable.*

Additive Function

Theorem 4.

Let $f : \mathbb{R} \to \mathbb{R}$ be additive which is not continuous. Then

1. *the graph of f is dense in $\mathbb{R}^2$;*
2. *the image of any open interval under f is dense in $\mathbb{R}$.*

Theorem 5.

Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function. Then the image of any open interval is dense in $\mathbb{R}$.

These results reveal that additive functions are either very regular or extremely pathological.

Ulam's stability problem

S.M. Ulam [was a Polish Mathematician] in his famous lecture in 1940 to the Mathematics club of the University of Winsconsin (US) presented a number of unsolved problems. This is starting point of the theory of stability of functional equations.

One of the questions led to a new line of investigation, nowadays known the stability problems. He proposed the first stability problem concerning group homomorphisms.

Ulam's stability problem

Given a metric group $G(*, d)$, a number $\varepsilon > 0$ and a map $f : G \to G$ which satisfies the inequality

$$d(f(x * y), f(x) * f(y)) < \varepsilon \quad \text{for all } x, y \in G,$$

(f is called an ε-**automorphism** of G).

Does there exist an automorphism g of G and a constant $k > 0$, depending only on G, such that

$$d(g(x), f(x)) \leq k\varepsilon \quad \text{for all } x \in G \ ?$$

(g is called **automorphism** of G).

If the answer is affirmative, then we say that the Cauchy equation $f(x * y) = f(x) * f(y)$ is **stable**.

Stability problem solved by Hyers

In 1941, Hyers partially solved the problem in the framework of Banach spaces. He has shown that δ may be taken to be ε.

Due to the question of Ulam and the answer of Hyers, the stability of equations is called after their names. Later, a large number of papers and books have been published in connection with various generalizations of Hyers-Ulam theorem.

Ulam's stability problem in Banach space settings

Let us rephrase the Ulam's stability problem in Banach space settings.

Let X be a normed linear space, Y be a Banach space and let ε be a positive number. A mapping $f : X \to Y$ is called ε-**additive** if

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$. Then the problem may be stated as : Does there exist for each $\varepsilon > 0$, a $\delta > 0$ such that to each ε-additive function $f : X \to Y$ there corresponds an additive mapping $g : X \to Y$ satisfying the inequality

$$\|f(x) - g(x)\| \leq \delta, \quad \forall x \in X \ ?$$

Stability problem solved by Hyers

Hyers in 1941 proved the following result.

Theorem 6.

Let X be a normed linear space, Y be a Banach space and let ε be a positive number. Let $f : X \to Y$ be ε-additive, that is,

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$. Then the limit $g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for each $x \in X$, and $g : X \to Y$ is the unique additive function such that

$$\|f(x) - g(x)\| \leq \varepsilon, \quad \forall x \in X.$$

Moreover, if f is continuous at a point in X, then g is continuous everywhere in X.

Stability problem solved by Hyers

This pioneering result of D.H. Hyers can be expressed in the following way.

The Cauchy additive functional equation

$$f(x+y) = f(x) + f(y)$$

is **stable** for any pair of Banach spaces.

The function

$$(x, y) \mapsto f(x+y) - f(x) - f(y)$$

is called the **Cauchy difference** of the function. Functions with a bounded Cauchy difference are called **approximately additive functions.**

The sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is called the **Hyers-Ulam sequence.**

Generalization

There are two possible ways to generalize Theorem 6. One of the natural ways is to generalize the domain ; the other is to have different bounds on the right side of

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon.$$

It is possible to prove a stability result similar to Hyers for functions that are not necessarily having bounded Cauchy differences. Mainily using the technique of Hyers, Th.M. Rassias in 1978 gave the following generalization of Hyers's result.

Functions that are not necessarily having bounded Cauchy differences

Rassias in 1978 proved the following result.

Theorem 7.

Let X be a normed linear space and Y be a Banach space. Let ε be a positive number and $p \in [0, 1)$. Let $f : X \to Y$ be such that $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$ holds for all $x, y \in X$. Then there exists a unique additive mapping $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p$$

for every $x \in X$.

The above result is called the **Hyers-Ulam-Rassias stability.**

Some Observations

- Theorem 7 is not true for $p = 1$. In 1991, Z. Gajda gave a counter example that a similar stability result does not hold for $p = 1$. Another example was given by Th. M. Rassias and P. Semrl.
- In 1991, Z. Gajda solved the problem for $p > 1$, which was raised by Th.M. Rassias. Hence the Theorem 7 is true for all real p expect $p = 1$.
- $p = 0$ in Theorem 7 yields Theorem 6.

Functions that are not necessarily having bounded Cauchy differences

Gajda in 1991 proved the following result.

Theorem 8.

Let X be a normed linear space and Y be a Banach space. Let ε be a positive number and $p > 1$. Let $f : X \to Y$ be such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

holds for all $x, y \in X$. Then the limit $g(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$, and $g : X \to Y$ is the unique additive function such that

$$\|f(x) - g(x)\| \leq \frac{2}{2^p - 2}\varepsilon, \quad \forall x \in X.$$

Hyers-Ulam stability of operator equation

T. Miura at el. in 2003 introduced the notion of the Hyers-Ulam stability of a mapping between normed spaces as follows :

Definition 9.

Let X and Y be normed linear spaces and let f be a (not necessarily linear) mapping from X into Y. Let $y_0 \in R(f)$.
We say that the equation $f(x) = y_0$ has the Hyers-Ulam stability *if there exists a constant $K \geq 0$ with the following property:*
For every $\varepsilon \geq 0$ and $x \in X$ satisfying $\|f(x) - y_0\| \leq \varepsilon$, there exists an element $x_0 \in X$ such that $f(x_0) = y_0$ and $\|x - x_0\| \leq K\varepsilon$.

We call K a HUS constant for the equation $f(x) = y_0$.

Definition 10.

Let X and Y be normed linear spaces. A mapping $f : X \to Y$ (not necessarily linear) has the Hyers-Ulam stability *if there exists a constant $K \geq 0$ with the following property:*
For every $\varepsilon \geq 0, y \in R(f)$ and $x \in X$ satisfying $\|f(x) - y\| \leq \varepsilon$, there exists an element $x_0 \in X$ such that $f(x_0) = y$ and $\|x - x_0\| \leq K\varepsilon$.

We call K a HUS constant for f and denote by K_f the infimum of all HUS constants for f. If, in addition, K_f becomes a HUS constant for f, then we call it the HUS constant for f.

By definition, if f has Hyers-Ulam stability, then so does the equation $f(x) = y_0$ for every $y_0 \in R(f)$.

Hyers-Ulam stability of a linear mapping between two normed linear spaces

Roughly speaking, if f has the Hyers-Ulam stability, then there exists a constant $K > 0$ with the property:

Fix $y \in R(f)$, to each "ε-approximate solution" x of the equation $f(x) = y$, there corresponds an exact solution x_0 of the equation in a $K\varepsilon$-neighborhood of x.

Hyers-Ulam stability of a linear mapping between two normed linear spaces

When we consider bounded linear operator, we can see that the Hyers-Ulam stability is closely related to the open mapping theorem.

If f is linear, the definition is equivalent to : To each ε-approximate solution x of the equation $f(x) = 0$, there corresponds an exact solution x_0 of the equation in the $K\varepsilon$-neighbourhood of x.

That is, for any $\varepsilon > 0$ and $x \in X$ with $\|f(x)\| \leq \varepsilon$, there exists $x_0 \in X$ such that $f(x_0) = 0$ and $\|x - x_0\| \leq K\varepsilon$. In other words, for any $x \in X$, there exists $x_0 \in N(f)$ such that $\|x - x_0\| \leq K\|f(x)\|$.

Hyers-Ulam stability of a bounded linear operator between two Banach spaces

Let X and Y be Banach spaces and T be a bounded linear operator from X into Y. Define the induced one-to-one linear operator $\tilde{T}$ from the quotient Banach space $X/N(T)$ into Y by

$$\tilde{T}(x + N(T)) = Tx, \forall x \in X.$$

Theorem 11.

Let X and Y be Banach spaces and T be a bounded linear operator from X into Y. Then the following statements are equivalent :

1. *T has the Hyers-Ulam stability.*
2. *T has closed range.*
3. *$\tilde{T}^{-1}$ from $R(T)$ onto $X/N(T)$ is bounded.*

Moreover, if one of (hence all of) the conditions (i), (ii), and (iii) is true, then $K_T = \|\tilde{T}^{-1}\|$.

Hyers-Ulam stability of a closed linear operator between two Hilbert spaces

Let T be a closed operator from a Hilbert space H into K with domain $D(T)$. We say that T is lower semibounded if there exists a constant $\gamma > 0$ such that

$$\|Tx\| \geq \gamma\|x\| \quad \forall x \in C(T).$$

Here $C(T) := D(T) \cap N(T)^{\perp}$, and it is called the carrier of T.

The reduced minimum modulus of T is defined by

$$\begin{aligned} \gamma(T) &= \inf_{\|x\|=1} \left\{ \|Tx\| : x \in C(T) \right\} \\ &= \sup_{x \in C(T)} \left\{ \gamma > 0 : \|Tx\| \geq \gamma\|x\| \right\}. \end{aligned}$$

Hyers-Ulam stability of a closed linear operator between two Hilbert spaces

Theorem 12.

Let T be a closed operator from H into K with domain $D(T)$. If T is lower semibounded, then T has the Hyers-Ulam stability with a HUS constant $\gamma(T)^{-1}$.

Let T be a closed operator from H into K with domain $D(T)$. Define $\hat{T} : C(T) \subseteq H \to K$ by

$$\hat{T}x = Tx, \forall x \in C(T).$$

Since T is closed, so is $\hat{T}$.

Hyers-Ulam stability of a closed linear operator between two Hilbert spaces

Theorem 13.

Let T be a closed operator from H into K with domain $D(T)$. Then the following are equivalent:

1. *T has the Hyers-Ulam stability.*
2. *T has closed range.*
3. *T^{-1} is bounded.*
4. *$\hat{T}^{-1}$ is lower semibounded.*

Moreover, one of the conditions above is true, then we have

$$K_T = \|\hat{T}^{-1}\| = \gamma(T)^{-1}.$$

Corollary 14.

Let T be a closed operator from H into K with domain $D(T)$. If T has the Hyers-Ulam stability, then K_T is the HUS constant for T.

Hyers-Ulam stability of a closed linear operator between two Hilbert spaces

We recall that every closed operator $T : D(T) \subset H \to K$ can be regarded as a bounded operator by considering graph inner product on $D(T)$.

Let $H_0 = (D(T), \text{graph norm})$. Hence H_0 is a Hilbert space. Define $T_0 : H_0 \to H$ by $T_0 x = Tx$, for all $x \in H_0$.

Then T_0 is a well-defined bounded operator. Next, we are concerned with the Hyers-Ulam stability of T_0. Moreover, we describe the HUS constant K_{T_0}.

Hyers-Ulam stability of a closed linear operator between two Hilbert spaces

Theorem 15.

Let T be a closed operator from H into K with domain $D(T)$. Then the following are equivalent:

1. *T has the Hyers-Ulam stability.*
2. *T_0 has the Hyers-Ulam stability.*

Moreover, if one of the constants (i) and (ii) is true, then the HUS constants K_T, K_{T_0} and the lower bounds $\gamma(T), \gamma(T_0)$ are connnected with the following relations:

$$\begin{aligned} K_T &= \gamma(T)^{-1} \\ K_{T_0} &= \gamma(T)^{-1} \\ K_{T_0}^2 &= K_T^2 + 1. \end{aligned}$$

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